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Lie Centrally Metabelian Group Rings

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Let $L(R)$ denote the associated Lie ring of an associative ring R under the Lie multiplication $[x, y] = xy - yx$; x, $y \in R$. Then R is said to be Lie centrally metabelian if the Lie ring $L(R)$ is centrally metabelian, that is, $[[x_1, x_2], [x_3, x_4], x_5] = 0$ for all $x_1, x_2, x_3, x_4, x_5 \in R$. Let G be a group and K be a field such that the group ring KG is Lie centrally metabelian. If char $K = 0$, then it is known that G is abelian. In this paper we have obtained two results on Lie centrally metabelian group rings, when char $K = p \geqslant 3$. The first states that if char $K = p > 3$, then KG is Lie centrally metabelian if and only if G is abelian. The second result proves that if char $K = 3$ and KG is Lie centrally metabelian then the commutator subgroup G' is a finite 3-Engel 3-group of exponent at most 9 and consequently nilpotent of class at most 4. Further it has been shown that if KG is Lie solvable and char $K = p$ is sufficiently large, then G is abelian. At the end, we obtain an interesting result which shows that $\delta^2(U(R)) - 1 \subseteq \delta^{(2)}(L(R))R$ for any ring R, where $\delta^{(2)}(U(R))$ and $\delta^{(2)}(L(R))$ denote the second derived term of the unit group $U(R)$ and the associated Lie ring $L(R)$ of R, respectively. As a consequence the unit group $U(R)$ is metabelian if the Lie ring $L(R)$ is metabelian. © 1992 Academic Press, Inc.

1. INTRODUCTION

Throughout ring will mean an associative ring with identity $1 \neq 0$. Any ring R may be treated as a Lie ring under the Lie multiplication $[x, y] = xy - yx$, $x, y \in R$. The Lie ring, thus obtained, will be denoted by $L(R)$ and will be called the associated Lie ring of R. In general, inductively, $[x_1, x_2, ..., x_n] = [[x_1, x_2, ..., x_{n-1}], x_n]$ and $[x, {}_n y] = [[x, {}_{n-1} y], y]$.

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Also a Lie ideal of R means an ideal of the Lie ring $L(R)$. Thus V is a Lie ideal of R if V is an additive subgroup of R and $v \in V$, $r \in R$ implies $[v, r] \in V$. The identity $vr = [v, r] + rv$ gives that $VR = RV$ for any Lie ideal V of R. Thus VR and RV are two sided ideals. For any two Lie ideals V and W of R, we shall denote by $[V, W]$ the additive subgroup of R generated by $\{[v, w] \mid v \in V, w \in W\}$ and it may be noted that $[V, W]$ is a Lie ideal.

The lower central and the derived chains of a Lie ideal V are defined inductively as $\gamma_1(V) = V$, $\gamma_{n+1}(V) = [\gamma_n(V), V]$ and $\delta^{(0)}(V) = V$, $\delta^{(n)}(V) = [\delta^{(n-1)}(V), \delta^{(n-1)}(V)]$, respectively. The ring R is called Lie solvable of length n (Lie nilpotent of class c) if $\delta^{(n)}(L(R)) = 0$ and $\delta^{n-1}(L(R)) \neq 0$ ($\gamma_{c+1}(L(R)) = 0$ but $\gamma_c(L(R)) \neq 0$).

If $\delta^{(2)}(L(R)) = 0$, then R is called Lie metabelian. R is said to be Lie centrally metabelian if $\delta^{(2)}(L(R))$ is contained in the centre of R, i.e., $[\delta^{(2)}(L(R)), R] = 0$.

Let G be a group written multiplicatively. For elements $g_1, g_2, \ldots, g_n$ in G, the commutators are defined by $(g_1, g_2) = g_1^{-1} g_2^{-1} g_1 g_2$ and inductively $(g_1, g_2, \ldots, g_n) = ((g_1, g_2, \ldots, g_{n-1}), g_n)$. The lower central chain and the derived chain of G are defined by $\gamma_1(G) = G$, $\gamma_{n+1}(G) = (\gamma_n(G), G)$ and $\delta^{(0)}(G) = G$, $\delta^{(n)}(G) = (\delta^{(n-1)}(G), \delta^{(n-1)}(G))$ respectively. We shall denote by G' the commutator subgroup $\gamma_2(G) = \delta^{(1)}(G)$ of G.

Let G be a group and K be a field. We denote by KG the group algebra of the group G over the field K. In this paper, we investigate as to when the group algebra KG is Lie centrally metabelian. Lie metabelian group rings were studied in [7]. Lie solvable and Lie nilpotent group algebras were characterized in [5]. If p is a prime, we say that G is p-abelian if G' is a finite p-group. G is o-abelian means G is abelian. If char $K = p \geqslant 0$ and $p \neq 2$, then KG is Lie solvable if and only if G is p-abelian and KG is Lie nilpotent if and only if G is p-abelian and nilpotent (see [8, V.3.1 and V.4.4]).

In Section 2, we study the group algebra KG when char $K = p > 3$ and in Section 3 we deal with char $K = 3$. Our methods do not work in char $K = 2$. In Section 4, we show that $\delta^{(2)}(U(R)) - 1 \subseteq \delta^{(2)}(L(R))\, R$ for an arbitrary ring R, where $U(R)$ denotes the group of all units in the ring R.

This work is continuation of the earlier work by the authors [9, 10, 11], and papers [1, 4, 7] have been very useful during the investigation.

2. Lie Centrally Metabelian Group Rings of Characteristic $p > 3$

Let R be a ring. We shall denote by J the ideal of R generated by all elements $[[r_1, r_2], [r_3, r_4], r_5]$ as r_1, r_2, r_3, r_4, r_5 vary over all elements of

R, i.e., $J=[\delta^{(2)}(L(R)), L(R)]\,R=R[\delta^{(2)}(L(R)), L(R)]$. Clearly R is Lie centrally metabelian if and only if $J=(0)$. Also we define I to be the ideal of R generated by all elements $[[r_1, r_2], [r_3, r_4], [r_5, r_6]]$, $r_i \in R$. Thus $I \subseteq J$ and $I=(0)$ in any Lie centrally metabelian ring.

In what follows, we shall repeatedly use the Lie identities $[xy, z]=x[y, z]+[x, z]\,y$, $[x, yz]=y[x, z]+[x, y]\,z$ and the Jacobi identity $[[x, y], z]+[[y, z], x]+[[z, x], y]=0$. Also wherever x^{-1}, y^{-1}, etc. occur, it is understood that x, y belong to the unit group $U(R)$.

LEMMA 2.1. *Let R be a Lie centrally metabelian ring such that $2 \in U(R)$ and let $x, y \in R$. Then*

(i) $[x, y][x, y, y]$ *and* $[x, y, y][x, y]$ *are central in* R;

(ii) $[[x, y]^3, y]=0$;

(iii) $[x, y]^2\,[x, y, y]=0=[x, y, y][x, y]^2$.

Proof. (i) $[[x, y], [xy, y]] = [[x, y], [x, y]\,y] = [x, y][x, y, y] \in \delta^{(2)}(L(R))$, which is central in R. Similarly $[x, y, y][x, y]=[[x, y], [yx, y]]$ is central.

(ii) and (iii).

$$
\begin{aligned}
[[x, y]^3, y] &= [x, y]^2\,[x, y, y]+[[x, y]^2, y][x, y] \\
&= [x, y]^2\,[x, y, y]+[x, y][x, y, y][x, y]+[x, y, y][x, y]^2 \\
&= 3[x, y]^2\,[x, y, y], \qquad \text{using (i)}.
\end{aligned}
$$

By opening the first term on the right, we get

$$[x, y]^3=[[x, xy], [yx, y]]-x[x, y, y][x, y]+y[y, x, x][y, x].$$

So $[[x, y]^3, y]=-[x, y]^2\,[x, y, y]$, by (i).

From above we see that $4[x, y]^2\,[x, y, y]=0$ and hence $[x, y]^2\,[x, y, y]=0$, as $2 \in U(R)$. We also get $[[x, y]^3, y]=0$. This proves (ii) and (iii).

Remark. If R is as in Lemma 2.1 and y^{-1} exists, then $[x, y][x, y, y^{-1}]$ and $[x, y, y^{-1}][x, y]$ are central and $[[x, y]^3, y^{-1}]=0$. Also expanding the middle term in $[x, y][x, y, y^{-1}y][x, y]=0$ and using (i) and (iii) above, we get

$$[x, y]^2\,[x, y, y^{-1}]=0=[x, y, y^{-1}][x, y]^2.$$

LEMMA 2.2. *In any ring R, we have*

(i) $2[x_1, x_2, y][y, z][x_1, x_2, y] \in I$;

(ii) $2[x_1, x_2, y][y^{-1}, z][x_1, x_2, y] \in I$.

Proof. (i) By expanding the first term on the right and adjusting, we have

$$\begin{aligned}2[x_1, x_2, y][y, z][x_1, x_2, y] = & -[[yzy, y], [x_1, x_2], [x_1, x_2]] \\ & + [[yz, y], [x_1, x_2], [x_1, x_2]]\, y \\ & + y[[zy, y], [x_1, x_2], [x_1, x_2]] \\ & - y[[z, y], [x_1, x_2], [x_1, x_2]]\, y.\end{aligned}$$

Clearly the right hand side belongs to I.

(ii) By part (i), $2[x_1, x_2, y^{-1}][y^{-1}, yzy][x_1, x_2, y^{-1}] \in I$.

By taking y out on the left from the middle term and adjusting with the first, we see that $2[x_1, x_2, y][y^{-1}, zy][x_1, x_2, y^{-1}] \in I$.

Similarly doing on the right, we get $2[x_1, x_2, y][y^{-1}, z][x_1, x_2, y] \in I$.

LEMMA 2.3. *If R is a Lie centrally metabelian ring in which $2 \in U(R)$, then $[x, y]^6 = 0$ for all $x, y \in R$.*

Proof. In the proof of Lemma 2.1 (ii), we have observed that $[x, y]^3 = z_1 - xz_2 + yz_3$ where $z_1 = [[x, xy], [yx, y]]$, $z_2 = [x, y, y][x, y]$ and $z_3 = [y, x, x][y, x]$ are central in R. By Lemma 2.1, it is easy to see that $z_1 z_2 = z_2 z_3 = z_1 z_3 = 0$ and $z_2^2 = z_3^2 = 0$. Also by Lemma 2.2 and Lemma 2.1 it can be seen that $z_1^2 = 0$, since $I = (0)$ and $2 \in U(R)$. We, therefore, have $[x, y]^6 = 0$.

LEMMA 2.4. *For all $a, b \in U(R)$ and $r \in R$,*

(i) $2[(a, b), b][b, r][(a, b), b] \in I$;

(ii) $2[(a, b), b][b^{-1}, r][(a, b), b] \in I$;

(iii) $2((a, b, b) - 1)^3 \in I$.

Proof. (i) $[(a, b), b] = [a^{-1}b^{-1}ab - 1, b] = [[a^{-1}, b^{-1}a]\, b, b] = [a^{-1}, b^{-1}a, b]\, b$ and $b[b, r] = [b, br]$.

The rest follows by Lemma 2.2(i).

(ii) Follows similarly by Lemma 2.2(ii).

(iii) $2((a, b, b) - 1)^3 = 2(a, b)^{-1} b^{-1}[(a, b), b][(a, b)^{-1}, b^{-1}](a, b)\, b[(a, b)^{-1}, b^{-1}](a, b)\, b = -2(a, b)^{-1} b^{-1}[(a, b), b][b^{-1}, (a, b)^{-1}]\, [(a, b), b](a, b)^{-1} b^{-1}(a, b)\, b$.

Thus $2((a, b, b) - 1)^3 \in I$, by part (ii), as desired.

We now obtain the main results of this section.

THEOREM 2.5. *Let K be a field of characteristic $p>3$ and G be a group such that the group algebra KG satisfies the following identity*:

$$[[x_1, x_2], [x_3, x_4], [x_5, x_6]] = 0 \text{ for all } x_1, x_2, x_3, x_4, x_5, x_6 \text{ belonging to } KG.$$

Then G is nilpotent of class atmost 2.

Proof. By Lemma 2.4(iii), $((g, h, h)-1)^3=0$, for all g, $h \in G$. Clearly $\delta^{(3)}(L(KG))=0$, so KG is Lie solvable. Therefore G' is a finite p-group by [8, V.3.1]. But $p>3$, hence $(g, h, h)=1$. Thus G is 2-Engel. It follows from ([6, Theorem 7.14], or [3]) that $(G', G)^3=1$. Again, since $p>3$ and G' is a p-group, we have $r_3(G)=(G', G)=1$. So G is nilpotent of class almost 2, as desired.

THEOREM 2.6. *Let K be a field of characteristic $p=0$ or $p>3$ and G be a group. Then KG is Lie centrally metabelian if and only if G is abelian.*

Proof. Clearly if G is abelian, then KG is Lie centrally metabelian, in fact commutative, in all characteristic. Also if char $K=0$ and KG is Lie centrally metabelian, then by ([8, V.3.1] or [9]) G is abelian.

Suppose, now, that char $K=p>3$ and KG is Lie centrally metabelian. By Theorem 2.5, G is nilpotent of class atmost 2. Thus G' is central. Also G' is a finite p-group by [8, V.3.1]. Now from Lemma 2.1 and the fact that (x, y) is central for all $x, y \in G$, we have

$$\begin{aligned}
0 &= [x, y]^2\,[x, y, y] \\
&= [x, y]^3\, y - [x, y]^2\, y[x, y] \\
&= \{yx((x, y)-1)\}^3\, y - \{yx((x, y)-1)\}^2\, yyx((x, y)-1) \\
&= \{(yx)^3\, y - (yx)^2\, y^2x\}\,((x, y)-1)^3, \\
&= (yx)^2\, y^2x\,((x, y)-1)^4.
\end{aligned}$$

This gives that $((x, y)-1)^4=0$. But G' is a p-group and $p>3$, so $(x, y)=1$. We conclude that G is abelian and the proof is complete.

If KG is Lie solvable and char $K=p>3$, then by [9, Theorem 2.1] the ideal J of KG is nilpotent. Let N be the nilpotency index of J. We deduce that for large p, Lie solvability of KG implies G is abelian.

THEOREM 2.7. *Let KG be Lie solvable and char $K=p>3$. If $p-1 \geqslant 4N$, then G must be abelian.*

Proof. Let g, $h \in G$. By Lemma 2.4(iii) $((g, h, h)-1)^3 \in I \subseteq J$ and this implies that $((g, h, h) - 1)^{3N} \in J^N = (0)$. Now $p - 1 \geqslant 4N$, so $((g, h, h) - 1)^{p-1} = 0$. But G' is a finite p-group [8, V.3.1], so $(g, h, h) = 1$. Thus G is 2-Engel. Now exactly as in the proof of Theorem 2.6, we have G' is central and for all $x, y \in G$, $((x, y) - 1)^4 = x^{-1}y^{-2}(yx)^{-2}[x, y]^2[x, y, y]$. Now KG/J is Lie centrally metabelian of characteristic $p > 3$, so by Lemma 2.1 $((x, y) - 1)^4 \in J$. Thus $((x, y) - 1)^{4N} \in J^N = (0)$. Again since $p - 1 \geqslant 4N$, we get $((x, y) - 1)^{p-1} = 0$ and $(x, y) = 1$. Hence G must be abelian.

3. Lie Centrally Metabelian Group Rings of Characteristic 3

Let $R = Z_3S_3$ be the group algebra of the symmetric group S_3 over the field Z_3 of integers modulo 3. It is not difficult to see that Z_3S_3 is Lie centrally metabelian but not Lie metabelian. Here S_3 is non-abelian and metabelian. Thus the situation in characteristic 3 is different from the characteristic $p > 3$, where Lie centrality of KG implies G is abelian.

In what follows, we shall repeatedly use Lemma 2.1 and the fact that $[x, y][x, y, y^{-1}]$ and $[x, y, y^{-1}][x, y]$ are central and $[x, y]^2[x, y, y^{-1}] = [x, y][x, y, y^{-1}][x, y] = [x, y, y^{-1}][x, y]^2 = 0$ in any Lie centrally metabelian associative algebra R over a field K of characteristic 3. This is remarked immediately after the proof of Lemma 2.1.

Lemma 3.1. *Let R be a Lie centrally metabelian associative algebra over a field K of characteristic* 3 *and let $x, y \in U(R)$. Then*

(i) $[(x, y) - 1, [x, y]^2] = 0$;

(ii) $[(x, y) - 1, [x, y]^3] = 0$;

(iii) $[(x, y) - 1, [x, y]]((x, y) - 1) = 0$.

Proof.

(i)
$$\begin{aligned}
&[(x, y) - 1, [x, y]^2] \\
&= [x^{-1}y^{-1}[x, y], [x, y]^2] \\
&= [x^{-1}y^{-1}, [x, y]^2][x, y] \\
&= x^{-1}[y^{-1}, [x, y]^2][x, y] + [x^{-1}, [x, y]^2]y^{-1}[x, y] \\
&= -x^{-1}[x, y, y^{-1}][x, y]^2 - x^{-1}[x, y][x, y, y^{-1}][x, y] \\
&\quad + [x, y][y, x, x^{-1}]y^{-1}[x, y] + [y, x, x^{-1}] \\
&\quad \times [x, y]y^{-1}[x, y] \\
&= 0, \qquad \text{as observed above.}
\end{aligned}$$

(ii) By Lemma 2.1, $[x, y]^3$ commutes with x and y and hence with $(x, y)-1$.

(iii)
$$\begin{aligned}
&[(x, y)-1, [x, y]]\,((x, y)-1)\\
&= [x^{-1}y^{-1}[x, y], [x, y]]\, x^{-1}y^{-1}[x, y]\\
&= -x^{-1}[x, y, y^{-1}][x, y]\, x^{-1}y^{-1}[x, y] + [y, x, x^{-1}]\\
&\quad \times y^{-1}[x, y]\, x^{-1}y^{-1}[x, y]\\
&= [[y, x, x^{-1}], [y^{-1}x, y]]\, x^{-1}y^{-1}[x, y],
\end{aligned}$$

other terms being 0,

$$\begin{aligned}
&= x^{-1}y^{-1}[[y, x, x^{-1}][x, y], [y^{-1}x, y]]\\
&\quad - x^{-1}y^{-1}[y, x, x^{-1}][[x, y], [y^{-1}x, y]]\\
&= 0,
\end{aligned}$$

since $[y, x, x^{-1}][x, y]$ and $[[x, y], [y^{-1}x, y]]$ are central.

LEMMA 3.2. *Let R and elements x, y be as in Lemma* 3.1. *Then*

$$((x, y)-1)^6 = 0.$$

Proof. By Lemma 2.3, we have $[x, y]^6 = 0$. Thus $x^{-1}y^{-1}[x, y][x, y]^5 = 0$ and we have $((x, y)-1)[x, y]^5 = 0$.

By Lemma 3.1 $[x, y]^5\,((x, y)-1) = 0$ and therefore

$$\begin{aligned}
x^{-1}y^{-1}[x, y][x, y]^4\,((x, y)-1) &= ((x, y)-1)[x, y]^4\,((x, y)-1)\\
&= [x, y]^4\,((x, y)-1)^2 = 0.
\end{aligned}$$

Continuing this way and using Lemma 3.1, we reach $((x, y)-1)[x, y]\,((x, y)-1)^4 = 0$, which upon using Lemma 3.1(iii) gives $x^{-1}y^{-1}[x, y]\,((x, y)-1)^5 = ((x, y)-1)^6 = 0$.

LEMMA 3.3. *Let the group algebra KG be Lie centrally metabelian and* char $K = 3$.

Then $(x, y)^3 = 1$ for all x, $y \in G$.

Proof. Since KG is Lie solvable and char $K = 3$, the commutator subgroup G' is a finite 3-group [8, V.3.1]. By Lemma 3.2

$$((x, y)-1)^6 = (x, y)^6 + (x, y)^3 + 1 = 0$$

and hence $(x, y)^3 = 1$, as desired.

It may be noted that in the situation of Lemma 3.3, $((x, y)-1)^3 = 0$ and thus $((x, y)-1)^2\,(x, y) = ((x, y)-1)^2$ for all x, $y \in G$.

LEMMA 3.4. *If the group algebra KG is Lie centrally metabelian and* char $K=3$, *then*

$$((x,y,y)-1)^2\,((x,y,y,y)-1)=0 \text{ for all } x, y\in G.$$

Proof. Let $a=(x,y)$ and $b=y$. Then

$$\begin{aligned}((a,b)-1)^2 &= (a,b)^2+(a,b)+1\\ &= (b,a)-1+(a,b)-1, \qquad \text{since } (a,b)^3=1 \text{ and char } K=3.\\ &= -b^{-1}a^{-1}[a,b]+a^{-1}b^{-1}[a,b]\\ &= [a^{-1},b^{-1}][a,b].\end{aligned}$$

Also

$$((a,b)-1)^3=[a^{-1},b^{-1}][a,b][a^{-1},b^{-1}]\,ab=0$$

and therefore

$$[a^{-1},b^{-1}][a,b][a^{-1},b^{-1}]=0.$$

Finally

$$\begin{aligned}&((a,b)-1)^2\,((a,b,b)-1)\\ &=((a,b)-1)^2\,(a,b)\cdot(a,b)^{-1}\,b^{-1}[(a,b)-1,b]\\ &=[a^{-1},b^{-1}][a,b]\,b^{-1}[[a^{-1},b^{-1}]\,ab,b]\\ &=[a^{-1},b^{-1}][ab^{-1},b][a^{-1},b^{-1}]\,ab^2\\ &=[(y,x)-1,y^{-1}][(x,y)\,y^{-1},y][(y,x)-1,y^{-1}](x,y)\,y^2\\ &=[y^{-1}[x^{-1}y,x],y^{-1}][(x,y)\,y^{-1},y][y^{-1}[x^{-1}y,x],y^{-1}](x,y)\,y^2\\ &=y^{-1}[x^{-1}y,x,y^{-1}][(x,y)\,y^{-2},y][x^{-1}y,x,y^{-1}](x,y)\,y^2\\ &=0, \text{ by Lemma 2.2(ii)}.\end{aligned}$$

We now prove the main result of this section.

THEOREM 3.5. *Let G be a group and K be a field of characteristic 3 such that KG is Lie centrally metabelian. Then G' is a finite 3-Engel 3-group of exponent at most 9 and it is nilpotent of class atmost 4.*

Proof. By [8, V.3.1] G' is a finite 3-group and by Lemma 3.3 $(g,h)^3=1$ for all $g, h\in G$. Let $x, y\in G'$, we show that $(x,y,y,y)=1$. If $a=(x,y)$ and $b=y$, then by Lemma 3.4

$$((a,b)-1)^2\,((a,b,b)-1)=0.$$

So $\{(a,b)^2+(a,b)+1\}\{(a,b)^{-1}(a,b)^b-1\}=0$. If $(a,b)=1$, then $(a,b,b)=(x,y,y,y)=1$. If $(a,b)\neq 1$, then the above equation gives that either $(a,b)^{-1}(a,b)^b=1$ or $(a,b)(a,b)^b=1$. But $(a,b)^{-1}(a,b)^b=1$ means $(a,b,b)=1$. Also $(a,b)(a,b)^b=1$ implies $(a,b)^{-1}(a,b)^b=(a,b,b)=(a,b)$. This gives that $(a,{}_nb)=(a,b)$ for all $n\geqslant 1$. But G' is nilpotent, so $(a,{}_nb)=1$ for some n and $(a,b)=1$. Thus $(x,y,y,y)=(a,b,b)=1$ always and G' is 3-Engel. It follows from [6, p. 48] that G' is nilpotent of class atmost 4.

Further let $x, y\in G'$. Then

$$\begin{aligned}(x,y^3)&=(x,y)(x,y)^y(x,y)^{y^2}\\&=(x,y)^2(x,y,y)(x,y)(x,y,y^2)\\&=(x,y)^3(x,y,y)((x,y,y),(x,y))(x,y,y)(x,y,y)^y\\&=(x,y,y)^3(x,y,y,y),\text{ since }(x,y)^3=1\text{ and }\gamma_5(G')=1\\&=1.\end{aligned}$$

We have proved that $y^3\in\zeta(G')$, that is, $G'/\zeta(G')$ has exponent 3 if it is non-trivial. By [6, Theorem 7.14] $G'/\zeta(G')$ is 2-Engel and $(x,y,y)\in\zeta G')$ for all $x, y\in G'$.

Next, for all $x, y\in G'$, it is easy to see that

$$\begin{aligned}(xy)^3&=y^3x^3x^{-1}(x,y)^2(x,y,y)\,x(x,y)\\&=y^3x^3(y,x)(y,x,x)(x,y,y)(x,y)\\&=y^3x^3(y,x,x)(x,y,y).\end{aligned}$$

Now any element of G' is $w=(x_1,y_1)(x_2,y_2)\cdots(x_n,y_n)$. If $n=1$, then $w^3=1$. If $n\geqslant 2$, then $w=w_1w_2$, where $w_1=(x_1,y_1)(x_2,y_2)\cdots(x_{n-1},y_{n-1})$ and $w_2=(x_n,y_n)$. We shall show by induction on n that $w^9=1$. From above $w^3=w_2^3w_1^3(w_2,w_1,w_1)(w_1,w_2,w_2)$ and therefore $w^9=w_1^9(w_2,w_1,w_1)^3(w_1,w_2,w_2)^3=1$. Thus exponent of G' is atmost 9. This completes the proof.

4. Lie Metabelian Rings

In this section, we consider Lie metabelian rings and their group of units. We use $\delta^{(n)}(L(R))$ for the nth term of the derived series of the associated Lie ring $L(R)$ (see Section 1 for definition). Our main result is the following theorem for $n=2$.

THEOREM 4.1. *If $U(R)$ denotes the group of units of a ring R, then*

$$\delta^{(n)}(U(R))-1\subseteq\delta^{(n)}(L(R))\,R \qquad \text{for} \quad n=0,1,2.$$

Proof. For $n=0$, the result is obvious. Also for $n=1$, it follows on the standard lines.

In order to prove $\delta^{(2)}(U(R))-1\subseteq\delta^{(2)}(L(R))\,R$, it is enough to prove that

$$((x,y),(u,v))-1\in\delta^{(2)}(L(R))\,R \qquad \text{for all} \quad x,y,u,v\in U(R).$$

It may be noted that $\delta^{(2)}(L(R))\,R=R\,\delta^{(2)}(L(R))$.

Further

$$((x,y),(u,v))-1=(x,y)^{-1}\,(u,v)^{-1}\,[(x,y),(u,v)].$$

Thus it suffices to show that $[(x,y),(u,v)]\in\delta^{(2)}(L(R))\,R$.

This we accomplice in two steps. First we prove that for any $b,c\in R$ and $u,v\in U(R)$, $[[b,c],(u,v)]\in\delta^{(2)}(L(R))\,R$. We proceed as follows:

$$\begin{aligned}
&[[b,c],(u,v)]\\
&\quad=[[b,c],u^{-1}v^{-1}[u,v]]\\
&\quad=u^{-1}v^{-1}[[b,c],[u,v]]+u^{-1}[[b,c],v^{-1}][u,v]\\
&\qquad+[[b,c],u^{-1}]\,v^{-1}[u,v]\\
&\quad=u^{-1}v^{-1}[[b,c],[u,v]]+u^{-1}[[b,c],[v^{-1}u,v]]\\
&\qquad-u^{-1}v^{-1}[[b,c],[u,v]]\\
&\qquad-[[b,c],[u,u^{-1}v^{-1}]]\,v+u^{-1}[[b,c],[u,v^{-1}]]\,v,\\
&\qquad\qquad \text{Using } v^{-1}[u,v]=-[u,v^{-1}]\,v
\end{aligned}$$

and taking $[u,v]$ with v^{-1} and $[u,v^{-1}]$ with u^{-1} inside the bracket.

$$\equiv 0 \bmod (\delta^{(2)}(L(R))\,R).$$

The second step uses exactly similar arguments on the first component (x,y) to get the following:

$$\begin{aligned}
&[(x,y),(u,v)]\\
&\quad=[x^{-1}y^{-1}[x,y],(u,v)]\\
&\quad=x^{-1}y^{-1}[[x,y],(u,v)]+x^{-1}[[y^{-1}x,y],(u,v)]\\
&\qquad-x^{-1}y^{-1}[[x,y],(u,v)]\\
&\qquad-[[x,x^{-1}y^{-1}],(u,v)]\,y+x^{-1}[[x,y^{-1}],(u,v)]\,y\\
&\quad\equiv 0 \bmod (\delta^{(2)}(L(R))\,R) \text{ by step one.}
\end{aligned}$$

The following is now immediate.

COROLLARY 4.2. *Let R be a ring which is Lie metabelian. Then the unit group U(R) is metabelian.*

Gupta and Levin [1] have proved that $\gamma_n(U(R)) - 1 \subseteq \gamma_n(L(R))\, R$ for all n. In view of Theorem 4.1 and [11, Lemma 1.4], perhaps, for a given n, there exists an m such that $\delta^{(n)}(U(R)) - 1 \subseteq \delta^{(m)}(L(R))\, R$. It may be helpful to note that $(\gamma_3(L(R))\, R)^2 \subseteq \delta^{(2)}(L(R))\, R$ [9, Corollary 1.10]. Also it is not known whether $(\delta^{(2)}(U(R)), U(R)) - 1 \subseteq [\delta^{(2)}(L(R)), L(R)]\, R$.

Rosenberger and Levin [7] have characterized Lie metabelian group rings. Some of these results can be obtained using the results of this paper. However, the results in [7] are obtained directly and in a straightforward manner.

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REFERENCES

1. N. GUPTA AND F. LEVIN, On Lie ideals of a ring, *J. Algebra* **81** (1983), 225–231.
2. H. HIENECKEN, Engelsche Elements der Lange drei, *Illinois J. Math.* **5** (1961), 681–707.
3. F. W. LEVI, Groups in which commutator operations satisfies certain algebraic conditions, *J. Indian Math. Soc.* **6** (1942), 87–97.
4. F. LEVIN AND S. K. SEHGAL, On Lie nilpotent group rings, *J. Pure Appl. Algebra* **37** (1985), 33–39.
5. I. B. S. PASSI, D. S. PASSMAN, AND S. K. SEHGAL, Lie solvable group rings, *Canad. J. Math.* **25** (1973), 748–757.
6. D. J. S. ROBINSON, "Finiteness Conditions and Generalized Soluble Groups, Part 2," Springer-Verlag, Berlin, 1972.
7. G. ROSENBERGER AND F. LEVIN, Lie metabelian group rings, Preprint No. 60, Ruhr-Universitat, Bochum, Dec., 1985.
8. S. K. SEHGAL, "Topics in Group Rings," Dekker, New York, 1978.
9. R. K. SHARMA AND J. B. SRIVASTAVA, Lie solvable rings, *Proc. Amer. Math. Soc.* **94** (1985), 1–8.
10. R. K. SHARMA AND J. B. SRIVASTAVA, Lie ideals in group rings, *J. Pure Appl. Algebra* **63** (1990), 67–80.
11. J. B. SRIVASTAVA AND R. K. SHARMA, "The Associated Lie Algebras of Group Algebras," pp. 187–197, Lecture Notes in Pure and Applied Mathematics, Vol. 91, Dekker, New York, 1984.